NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

VOL. 10 BATON ROUGE, LA., FEBRUARY, 1936 No. 5

PUBLISHED BY LOUISIANA STATE UNIVERSITY

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Published by the Louisiana State University Press

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May Be Had
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Subscription, $1.50, Net,
Per Year in Advance
Single Copies, 20c

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Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

All Business Communications should be addressed to the Editor and Manager, P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Universal Science

When one reflects that the applications of mathematics and of mathematical method have without an exception increased with the opening and development of every new field of science, the question easily arises, may we not assume as a philosophic principle that every substantial increase in thinking effectiveness and certainty of knowledge must be conditioned upon a corresponding increase in the use of mathematics or of mathematical method? Checking with an affirmative answer is the not altogether trivial fact of the lineage of the word "mathematics", its remote etymological ancestor being "Manthanein", which means "To learn". Consistent with the idea is the now near-classic language of Lord Kelvin (1883): "When you can measure what you are speaking about and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it with numbers, your knowledge is of a meagre and unsatisfactory kind". Harmonizing, is the historical citation of J. R. Partington in his "Higher Mathematics for Chemical Students": "Wengel and Richter, of mathematical temperament, laid the foundations of the 'art of measuring the chemical elements'; and Dalton, the mathematical tutor combined the whole of the results of quantitative measurement.... in a comprehensive theory based on the chemical atom". Corroborative, also, is a citation by the same writer of the public statement made by a scientist—(one which has doubtless been approved by uncounted others): "These ultimate laws—in the domain of the physical at least—will be the dynamical laws of the relation of matter to numbers, space and time themselves. When these relations shall be known, all physical phenomena will be a branch of pure mathematics."

A single example, taken from scores of instances in the history of science, well illustrates this INEVITABLENESS of mathematics as an instrument for the perfection of knowledge. We refer to the now familiar fact that Sir Humphry Davy showed heat to be, not a substance, "caloric" but a form of energy due to motion. Being a form of energy and hence susceptible of measurement, only by a mathematical formulation can a precise description of it be made.

But, it may be protested, is not there a knowledge unacquired by the process mathematical? The answer is: Merely to sense a thing or a quality is not of necessity to know all about it. So to know demands comparison, comparison requires measurement, measurement requires number.

S. T. SANDERS.

Mathematics As a Personal Experience*

By DEAN PAUL P. BOYD
University of Kentucky

Those who work in the field of mathematics, especially in the philosophical or purely theoretical phases of the subject, seem handicapped as compared with their co-workers in other fields more obviously related to the experiences of the "man in the street". They suffer somewhat the same detachment from affairs that a poet or a philosopher experiences. The practical man looks upon them as harmless yet useless members of society—at least useful only in a very remote sense. One is reminded of the busy Martha's protest to Jesus over Mary's detachment from the worries of the day's work, and of the practical Xantippe's exasperation because of Socrates' indifference to her realities.

Nevertheless the mathematician goes the serene way of his spiritual and mental life. He knows that the unseen world is not unrelated to the world of sense, that if there is any sense at all to life, if there is to be any answer to the Riddle of the Universe, if there is to be any real human progress they must come not from the children playing in the market place, nor from those industriously pulling brass tacks from the chairs and putting them back again, to use Ruskin's figure, but from the thinkers who deal with principles back of phenomena.

It is sometimes difficult, to convince the non-mathematical man, if there be such, that the life of the mathematical philosopher is well spent. One of the oldest of the "arts", made the basis by Plato and other philosophers of rational thinking, running through the centuries in a stream of ever-increasing volume, unlocking the doors of science and industry with golden keys, as pervasive in the structure of modern life as the air we breathe, mathematics yet is called upon to justify itself in education by "educators", and in practical affairs by "captains of industry."

But it is not my intention to write an apology for mathematics. By the discerning student of civilization she is admitted to an honorable and even pre-eminent place. He has found that even when "dead", she yet speaketh. He may say that it is all a clouded world to him, yet he is willing to believe that where there is so much smoke there must be considerable fire. Nor is it my plan to describe the immense modern vitality of this domain of thought by presenting the

*A paper presented to the Indiana Section of the Mathematical Association of America, Hanover, Indiana, May 4, 1935.

wide-spread extension of mathematics into fields long considered non-mathematical and the enormous amount of literature that is falling from the mathematical presses each succeeding year. Man, at his best, *is* mathematical, and he works out his destiny on the solid foundations of his own mathematical treads, and he flies toward the sun on his own mathematically-made wings, and he enriches his inner life with mathematical dreams, tenuous and fairy like, beautiful in their form and relationships, lovlier than e're were seen on land or sea.

My interest now is with mathematics rather as a matter of personal experience. Here is the supreme justification for devotion to the austere yet beautiful and generous goddess. In our own spiritual lives we experience certain blessings, because of our mathematical activities, that make life more worthwhile, richer and deeper. There are ideals, interpretations and great consolations.

There is the ideal of accuracy. Mathematics is nothing if not accurate. Its distinguishing characteristic is its insistence on accuracy in concepts, in definitions, in assumptions, in argument. Whenever the mathematician has erred in this respect others have risen to set him right. The story is replete with the eliminations of vagueness, the corrections of false assumptions, the completions of deductions, the refinements of processes. We looked at the sun with a smoked glass, then with small and imperfect lenses, then with powerful magnifiers, then with the photographic plate and the spectroscope, and the micrometer. The ideal has been accuracy to the limits of possibility. This is the first essential to arrival at truth.

There is next the ideal of logical perfection. The definition of mathematics has expanded with the science. Once mathematics was the science of magnitude and measurement; later it was the science of necessary conclusions; later still it has been enlarged to co-extensiveness with symbolic logic itself. The mathematician is bound by the limitations of logic. For him casuistry, sophism, prejudice, bias, rationalism, half-views, the personal equation are simply intolerable. Not that he ignores the fact and importance of the emotions. He is himself usually highly emotional, though it may be that he has his emotions under control, and he knows the relationship of emotion to will.

These ideals of accuracy and logic, really one ideal of accuracy, color the mathematicians thinking on all questions, both public and private. Here, incidentally, is one of the great reasons for the teaching of mathematics to the young. Transfer of training is still generally admitted to be a fact so far as transfer of ideals is concerned, and no

other subject of the curriculum offers such advantages for the development of these ideals of accuracy and logical soundness.

When the mathematical mind reads or hears a serious discussion he instinctively asks questions. Are the statements true? Are the assumptions at the base of the discussion justified? Are the arguments sound according to logical principles? Are all the facts stated and taken into consideration? Are all the consequences recognized? Is the man honest in his presentation? When the mathemical mind attacks a problem in law or medicine or business, he instinctively follows the same ideals—gets at the facts, reasons logically, so far as the problem is susceptible of logical treatment, to the necessary conclusions.

This habit of the mathematical mind is not altogether a comfortable nor a safe one to live with. He thinks that he sees the matter clearly in its logical and factual outlines, and he is impatient with others who do not. He is in danger of intolerance and egotism. The other man becomes a "bone-head" or a "fuzzy thinker" or a "crook", when he may not be any of these. He may designedly, with justification, for the sake of emphasis, or with the object of bringing more influence to bear on the emotions and the will, neglect that logical completeness so dear to the mathematician.

The mathematical mind is in danger also of underestimating the value of "clothes". To him the argument is the essential thing, with its conciseness and vigor and inevitableness. He may forget that beauty of expression and grace and eloquence and sympathy are great and powerful human interests and that if he wishes to bring his thinking to human use he must touch men not only through reason but also through their tastes and emotions.

In spite however,of his limiting tendencies the mathematician's ideal is that of the liberated mind. He wishes to live in a world where mind is free and to see the world whole. He does not wish to draw apart from the busy paths of human life but rather to bring the same ideals that he holds in his mathematical thinking to bear on the problems of humanity. If he can put his mathematical ideals to work in all the interests of life he is getting closer to truth, nearer a full understanding of things as they are, more effective in bringing to pass things as they should be.

This is the second great contribution of mathematics to life-understanding. The mathematician is incurably philosophical. He wonders about life, about the forces of nature, about God. What's it all about? He beats his thought-wings against his cage. He takes little flights of speculation, here and there, and wonders. He does

not despair of some day knowing more about reality than he now knows. The unknown is not necessarily the unknowable. He has found that new theories embrace and clarify older theories, that the unrolling scroll of truth shows wonderful unity.

The mathematician finds wherever he delves that this is an ordered world—and that the laws of thought that he has employed in his theorizing are universally significant. He finds, too, that his thought world is far more extensive than that of nature, so that all nature becomes but a small part of his world of relationships, a special case. He sees his thought encompassing the things of sense in an illimitable sphere. Why think the center of the sphere so real and the surrounding space so negligible? He includes an idealism in his philosophy. The ultimate reals he assigns to the world of spirit.

And in his own world of reals he discovers an ever expanding content. The microcosmus of magnitude and measurement has become a macrocosmus of concepts and systems, complete and independent in themselves yet all bound by the warp of self-consistency and the woof of the universal mathematical mode. Growth is a sign of life and of conformity with environment. The steady unfoldment of the mathematical organism through the centuries, the astounding expansion of the subject in recent years, all bring assurance, if any were needed, of the undying vitality, of the supreme naturalness of mathematical thinking to the mind of man. The mathematician then sees the universe in process of change, of ordered growth.

How natural such a world seems to the student of variables and functional relationships. Variables came into the mathematician's view long ago, doubtless suggested by natural phenomena. Functional relationships emerged. Now we have built them and classified them, and we see no end to the possibilities of discovery and invention in the functional field. We are treating them to processes of substitution and transformation, bringing to light new facts and new relationships. We find that certain functions remain invariant under transformation. We have learned that the most complex multiple points can be resolved into their simple elements through transformation. We have established correspondence between related forms. We have discovered a wonderful principle of duality running through our geometry. We have learned that the element composing a given form may be any one of several, and that dimensionality is a convention.

Such things suggest that in this variant universe there must be independence and dependence, variation and invariation, correspondences relating the complex to the simple, elements of various kinds,

varieties of viewpoints, new truths to be gotten from old by replacement, interpretation, relativity.

We might use as an illustration the idea of coordinates—many systems, all equally true, but differing in convenience for a particular task. We might use non-Euclidean geometry as an example of the suggestion—possibilities of mathematical concepts to the philosophically minded. We might bring out the speculative values of n—dimensional space. We might trace the development of the old philosophy of relativity through mathematical analysis to the present. We might turn again to simpler ideas like those of a limit and of infinity and point out how the mathematical mind clarifies and makes definite an idea that has elsewhere long been hazy. We might suggest the part that considerations of convenience play in mathematical procedure. It must suffice, however, to say that every great mathematical concept or process carries over into other fields of thought and suggests applications and analogies. Through them we gain insight into the structure of the cosmos, into the answers to the riddles of life, into the nature of God himself. A clearer appreciation of reality, a fuller understanding of the human show, these are the personal experiences of the mathematical mind.

But does this interpretation, this understanding satisfy? Is there an illuminating and comforting philosophy that grows out of it? Is there real joy in the pursuit of mathematical interests? Does it function in the making of a wholesome, a self-controlled, a confident, a large-minded, a big souled, a happy life? How can it be thought otherwise of this world of pure ideas, reaching down as it does into the realm of matter, bestowing as it has and will so many practical blessings to mankind in the making of life more comfortable and more efficient, yet reaching upward into the highest realms of speculation?

Here surely is the real justification for spending these previous days of our lives in the pursuit of what to some seem so useless. There is a plausible theory that all art originated in man's primitive spirit of play. In mathematics there is certainly this feeling of joyous abandon, of recreation, of freedom from annoying restrictions of time and place, of following where the spirit leads, of constructiveness. There is the appeal to man's artistic impulses—his playful imagination, his feeling of satisfaction in his work, and his love of the beautiful. Beauty is after all largely subjective. In his ethereal world the mathematician revels in the beauties of form and relationships, he walks as it were through landscapes of surpassing perfection; he mounts to the crest of the hill and breathes the pure clear invigorating air of its heights; he looks back over the way he has come and sees it in its

entirety; he looks forward to the surprises of the beauteous peaks still ahead, and wonders and aspires.

As,

"Thru some Helvetian Dell; when low hung mists
Break up and are beginning to recede;
How pleased he is where thin and thinner grows
The veil, or where it parts at once, to spy
The dark pines thrusting forth their spiky heads;
To watch the spreading lawns with cattle grazed;
Then to be greeted by the scattered huts
As they shine out; and see the streams whose murmur
Had soothed his ear while they were hidden; how pleased
To have about him which way e'er he goes
Something on every side concealed from view,
In every quarter something visible
Half seen or wholly, lost and found again,
Alternate progress and impediment,
And yet a growing prospect in the main."

A sense of detachment from pettiness and care fills his soul, comfort and peace are his. Not of an earthly valley and stream but of these streams and valleys of the land of his dreams would he say:

"Ah no, the stream
Is flowing, and will never cease to flow,
And I shall float upon that stream again.
By such forgetfulness the soul becomes,
Words cannot say how beautiful; then hail,
Hail to the visible Presence, hail to thee,
Delightful valley, habitation fair;
And to whatever else of outward form
Can give an inward help, can purify
And elevate, and harmonize, and soothe,
And steal away, and for a while deceive
And lap in lasting rest, and bear us on
Without desire in full complacency,
Contemplating perfection absolute
And entering as in a placid sleep."

Substituting "mathematics" for "nature", we may say with Wordsworth:

. . "Mathematics never did betray
The heart that loved her; 'tis her privilege,
Through all the years of this our life, to lead
From joy to joy; for she can so inform
The mind that is within us, so impress
With quietness and beauty, and so feed
With lofty thoughts, that neither evil tongues,
Rash judgments, nor the sneers of selfish men,
Nor greetings where no kindness is, nor all
The dreary intercourse of daily life,
Shall e'er prevail against us, or disturb
Our cheerful faith, that all which we behold
Is full of blessings."

". . there I found
Both elevation and composed delight:"

*"** there, recognized*
A type, for finite natures, of the one
Supreme Existence, the surpassing life
Which—to the boundaries of space and time,
Of melancholy space and doleful time,
Superior and incapable of change,
Nor touched by welterings of passion,—is,
And hath the name of God. Transcendent peace
And silence did await upon these thoughts
. . ."

". . Mighty is the charm
Of those abstractions to a mind beset
With images and haunted by herself,
And specially delightful unto me
Was that clear synthesis built up aloft
So gracefully; even then when it appeared
Not more than a mere plaything or a toy
To sense embodied: not the thing it is
In verity, an independent world,
Created out of pure intelligence."

Such may be, such *is* to a greater or less degree, the personal experience of the mathematician. Ideals, understanding, joy!

"*How charming is divine Philosophy!*"

"I shall detain you no longer ***, but straight conduct ye to a hillside, where I will point ye out the right path of a virtuous and noble education; laborious indeed at the first ascent but else so smooth, so green, so full of goodly prospect and melodious sounds on every side that the harp of Orpheus was not more charming."

Mathematical Meet in Hattiesburg March 13, 14

Mathematics and science teachers of Mississippi and Louisiana are to hear Professor J. O. Hassler in Hattiesburg March 13, 14. Professor Hassler is Professor of Mathematics and Astronomy at the University of Oklahoma, is retiring president of the National Council of Teachers of Mathematics, a member of the board of the Mathematical Association of America, and author of high school mathematics texts. He brings a message to high school teachers on Saturday morning at the Council session. Of especial interest is his Friday afternoon address on "An Evaluation and Comparison of Objectives and Subjective Tests in Mathematics". The high school and college teachers of the states are urged to attend.

Social features of the meeting include a tea at State Teachers College following the afternoon session and a banquet held at Mississippi Woman's College at seven p. m., Friday, March 13. At the banquet Professor Hassler will speak on "The Spirit of Discovery—a Motivating Power in Learning Mathematics and Science". The local committee on arrangements include Professor Dewey S. Dearman of State Teachers College, Dr. George A. Baker of Mississippi Woman's College, Professor J. A. Beeson of Hattiesburg High School. Professor O. V. Austin of State Teachers College is preparing the science program.

Other features of the program will include a paper on "Some Probelms of the Mathematics Teacher", by Professor R. L. O'Quinn of Louisiana State University, and a report by a committee on college and high school mathematics. The complete program is to be announced soon.

Professor Hassler is sent by the Mathematical Association of America but his interest is as great in high school teaching as in college teaching. He is author of many texts and magazine articles and is co-author of "The Teaching of Secondary Mathematics."

DOROTHY McCOY, Sec'y-Treas.

Note on Sums Involving Binomial Coefficients

By P. H. DAUS
University of California at Los Angeles

It is very well known that if in the binomial expansion of $(1+x)^n$, we replace x by 1 or -1, respectively, we find

$$2^n = 1 + C_1 + \ldots + C_r + \ldots + C_n,$$
$$0 = 1 - C_1 + \ldots \pm C_r \mp \ldots \pm C_n,$$

where C_r is the number of combinations of n things taken r at a time. By addition and subtraction of these relations, we find expressions involving alternate binomial coefficients. It is not quite so well known that if we replace x by a set of the kth complex roots of unity, we can find similar relations involving every kth coefficient. For example, if $\omega, \omega^2, 1$ are the cube roots of unity, then

$$(1+1)^n = 1 + C_1 + C_2 + \ldots,$$
$$(1+\omega)^n = 1 + \omega C_1 + \omega^2 C_2 + \ldots,$$
$$(1+\omega^2)^n = 1 + \omega^2 C_1 + \omega C_2 + \ldots,$$

and by addition of these relations, we find

$$1 + C_3 + C_6 + \ldots = \frac{2^n + (1+\omega)^n + (1+\omega^2)^n}{3}$$

a real integer. By means of the elementary properties of complex numbers, we find

$$1 + C_3 + C_6 + \ldots = \frac{2^n + 2\cos(n\pi/3)}{3},$$

which has a different numerical value, readily calculated, for any given n.

Similar relations may be found beginning with C_1 or C_2, and also relations in which the signs alternate.

An illustration of a sum of this type arises when we attempt to formally find the infinite series expansion for $e^x \sin x$ from the series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots,$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

By multiplying these together, we find

$$e^x \sin x = x + x^2 + x^3 \left(\frac{1}{2!} - \frac{1}{3!}\right) + x^4 \left(\frac{1}{3!1!} - \frac{1}{1!3!}\right) + \ldots,$$

with the coefficient of x^n being

$$a_n = \frac{1}{(n-1)!1!} - \frac{1}{(n-3)!3!} + \frac{1}{(n-5)!5!} - \ldots.$$

If we multiply and divide by n!, we can write

$$a_n/n! = C_1 - C_3 + C_5 - \ldots.$$

To evaluate this sequence, we place $x = i$ and $-i$ in $(1+x)^n$, subtract and divide by 2i, thus obtaining

$$\frac{a_n}{n!} = \frac{(1+i)^n - (1-i)^n}{2i} = (2)^{n/2} \sin(n\pi/4),$$

so that

$$e^x \sin x = x + \frac{2x^2}{2!} + \frac{2x^3}{3!} - \frac{2x^5}{5!} - \ldots + (2)^{n/2} \sin(n\pi/4) \frac{x^n}{n!}.$$

Theory of the Keuffel and Esser Logarithmic Spiral Curve

By H. R. GRUMMANN
Washington University Engineering School

The description of the curve contained in the booklet printed by the manufacturers of the curve makes no use of calculus. It is felt that many mathematics teachers and others who have had an elementary course in calculus would be more interested in the Keuffel and Esser Logarithmic Spiral Curve if they were supplied with a succinct account of its evolute property.

Let the equation of the curve in polar coordinates be written in the form

I. $$r = e^{\theta/m}$$

where (r, θ) are the polar coordinates of any point on the curve and m is a constant. The numerical value of m has been chosen so that the center of curvature of the point (r_1, θ_1) is itself a point on the curve with coordinates $(r_1/\tan\psi, \theta_1 - 270°)$. ψ is the constant angle from the radius vector at any point to the tangent line at that point.

From the calculus formula $\tan\psi = r/r'$ and equation I. we have

II. $$\tan\psi = m, \text{ a constant.}$$

By thinking of a right-angled triangle with acute angle ψ, and with opposite and adjacent legs of m and 1, respectively, one can easily write down the other trigonometric functions of ψ, as needed below.

From the calculus formula for the differential of arc length in polar coordinates, viz.,

$$dS = \sqrt{r^2 + r'^2}\, dt$$

we calculate dS for the polar graph of Equation I.:

$$dS = r \csc\psi \, d\theta$$

Integrating this we get

III. $$S = r \sec\psi + K$$

From the calculus formula for the curvature of a polar coordinate curve, viz.,

$$\text{curvature} = [(1/r) + (1/r)''] \sin^3 \psi$$

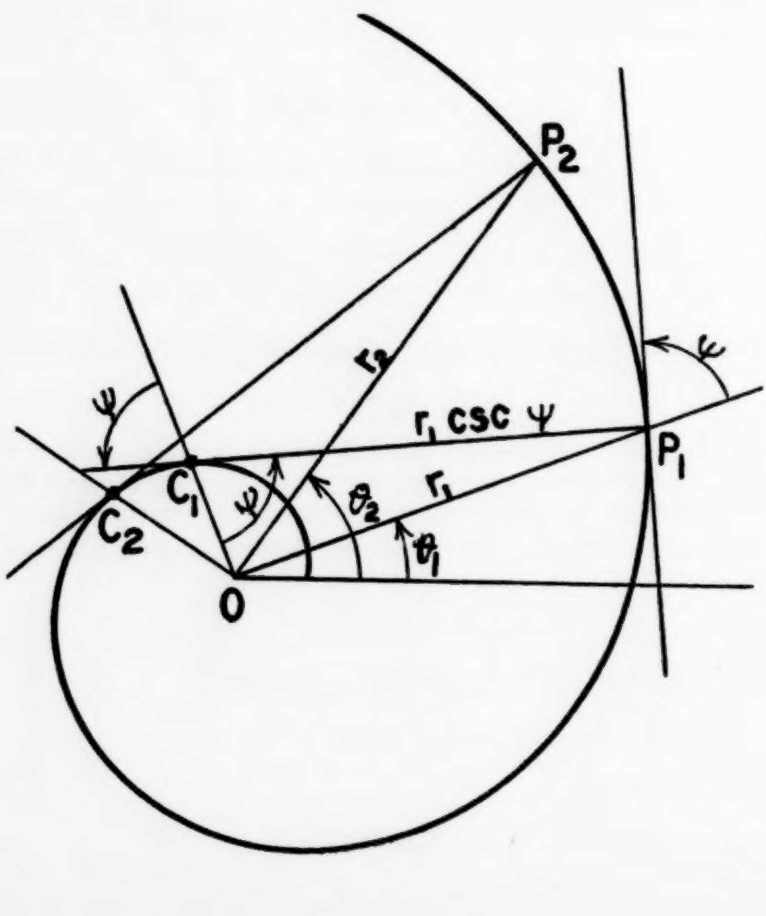
where $(1/r)''$ means the second derivative of the reciprocal of r with respect to θ, we have for the polar graph of Equation I.:

$$\text{curvature} = (\sin \psi)/r$$

and hence the radius of curvature R is given by

IV. $$R = r \csc \psi$$

In the figure $P_1 \equiv (r_1, \theta_1,)$ and $P_2 \equiv (r_2, \theta_2)$ are any two points on the curve.



The radius vector r_1, of the point (r_1, θ_1) is drawn and the corresponding center of curvature C_1 is also represented in the figure. C_1P_1 is the radius of curvature of the curve at the point P_1, and hence by Equation IV. has a length $r_1 \csc \psi$. Join the point C_1 with the pole 0. The Keuffel and Esser Logarithmic spiral is constructed from Equation I., using a numerical value of the constant m which makes the angle P_1OC_1 always a right angle. If angle $P_1OC_1 = 90°$, then by solving the triangle POC_1 we find that angle $OC_1P_1 = \psi$, and side $OC_1 = r_1/\tan \psi$. That C_1 is a point on the same logarithmic spiral as P_1 is then evident, since the vectorial angles of OC_1 and OP_1 differ by the constant angle 270° and $OP_1 = (OC_1) \tan \psi$. (If the vectorial angles of points on the graph of Equation I. increase in an arithmetical progression, the radius vectors of those points increase in a geometrical progression). That P_1C_1 is tangent to the spiral at the point C_1 follows from the fact that angle $OC_1P = \psi$. Hence, if angle $C_1OP_1 = 90°$, the spiral is its own evolute.

In this event, the difference of the radii of curvature at P_1 and P_2 is

$$R_2 - R_1 = (r_2 - r_1) \csc \psi,$$

using Equation IV. The coordinates of points C_1 and C_2 are respectively,

$$(r_1/\tan \psi,\ \theta_1 - 270°) \text{ and } (r_2/\tan \psi,\ \theta_2 - 270°)$$

Using equation III., the length of arc of the curve from C_1 to C_2 is $S_2 - S_1$

$$= (r_2 - r_1) \sec \psi / \tan \psi$$

$$= (r_2 - r_1) \csc \psi$$

$$= R_2 - R_1.$$

The numerical value of m that makes the angle P_1OC_1 a right angle may be calculated from the following consideration:

$$\frac{OP_1}{OC_1} = \tan \psi = [e^{\theta_1/m} \div e^{(\theta_1 - 270)/m}] = e^{3\pi/2m} = m.$$

Hence $$\log_e m = 3\pi/2m$$

The root of this transcendental equation may be readily obtained by successive approximations as follows:

m	$\frac{4.712}{m}$	$\log_e m$
3.5	1.348	1.252
3.7	1.272	1.308
3.6	1.310	1.280
3.65	1.290	1.295
3.642	1.294	1.292
3.644	1.293	1.293

Hence $\psi = \tan^{-1} m = \tan^{-1} 3.644 = 74°39'.2$, approximately.

On the Properties of a Determinant Function[1]

By CLIFFORD BELL
University of California at Los Angeles

Let the coordinates of any point of a plane curve be expressed in the form,

(1) $$x_1 : x_2 : x_3 = f_1(t) : f_2(t) : f_3(t),$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$ are functions of a parameter t, each of which is developable into a Taylor's series. Let a function, $F_i(t)$, be defined by the determinant, $|f_1(t)\ f_2'(t)\ f_3^{(i)}(t)|$, where the superscript (i) indicates that a derivative is to be taken with respect to the parameter t. It is evident under this definition that $F_1(t) \equiv 0$.

The function, $F_2(t)$, is the Wronskian of the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ with respect to t. It is also the well-known determinant[2] of higher plane curves which vanishes for those values of t that give inflections or cusps. $F_3(t)$, the derivative of $F_2(t)$, vanishes for those values of t that give cusps.

The general projective transformation, $\sigma x_j' = \sum_{k=1}^{3} a_{jk} x_k$, $(j = 1,2,3,)$ $\Delta = |a_{jk}| \neq 0$, applied to the equations of the curve (1) give for the new equations,

$$x_1' : x_2' : x_3' = \sum_{k=1}^{3} a_{1k} f_k(t) : \sum_{k=1}^{3} a_{2k} f_k(t) : \sum_{k=1}^{3} a_{3k} f_k(t).$$

Let $F_i'(t)$ represent the new $F_i(t)$ function. $F_i'(t)$, in determinant form, is $\left|\sum_{k=1}^{3} a_{1k} f_k(t)\ \sum_{k=1}^{3} a_{2k} f_k'(t)\ \sum_{k=1}^{3} a_{3k} f_k^{(i)}(t)\right|$, which, by the law of multiplication of determinants, reduces to the product $|a_{jk}| \cdot |f_1(t)$ $f_2'(t)\ f_3^{(i)}(t)|$ or $\Delta \cdot F_i(t)$. Thus $F_i(t)$ is invariant under the general projective transformation.

It is evident from the invariant nature of $F_i(t)$ that any property of the curve indicated by its vanishing must be a property independent of the triangle of reference used. Thus in investigating the effect of a cusp of a rational curve on $F_2(t)$, one may choose the triangle of refer-

[1]Presented to the American Mathematical Society, Dec. 2, 1933.
[2]See Hilton, Plane Algebraic Curves, *Oxford University Press*, 1920, P. 138.

ence in such a way that the equations of the curve are in the form, $x_1 : x_2 : x_3 = t^2g_1(t) : t^2g_2(t) : g_3(t)$, where $g_1(t)$ and $g_2(t)$ are polynomials of degree $n-2$ and $g_3(t)$ is a polynomial of degree n, the cusp being given by $t=0$. $F_2(t)$ for this curve is $At^5+Bt^4+Ct^3+Dt^2$, where A, B, C, D are functions of g_1, g_2, g_3 and their derivatives. Hence the parameter of the cusp appears as a double root of $F_2(t)=0$, which verifies known results.

If the curve (1) has a multiple point given by two or more equal values of the parameter t_1, it is known that $f_1(t_1)/f_1'(t_1) = f_2(t_1)/f_2'(t_1) = f_3(t_1)/f_3'(t_1)$. Thus the first two rows of the determinant $F_i(t_1)$ are proportional and hence $F_i(t_1)$ vanishes for all values of i.

In a proper parametric representation of a curve, to each ordinary point corresponds a single value of the parameter. To a multiple point corresponds more than one value of the parameter, two or more of which may be equal.

Let the equations (1) be a proper parametric representation of a curve to which the tangent line has $s-$point contact at (t_1), where (t_1) is either an ordinary point or a multiple point at which the parameter t_1 is different from all other parameters at that point. The equation of the tangent line is

$$x_1[f_2(t_1)f_3'(t_1) - f_2'(t_1)f_3(t_1)]$$
$$-x_2[f_1(t_1)f_3'(t_1) - f_1'(t_1)f_3(t_1)]$$
$$+x_3[f_1(t_1)f_2'(t_1) - f_1'(t_1)f_2(t_1)] = 0.$$

This line meets the curve at the points whose parameters are given by the equation

$$f_1(t)[f_2(t_1)f_3'(t_1) - f_2'(t_1)f_3(t_1)] \tag{2}$$
$$-f_2(t)[f_1(t_1)f_3'(t_1) - f_1'(t_1)f_3(t_1)]$$
$$+f_3(t)[f_1(t_1)f_2'(t_1) - f_1'(t_1)f_2(t_2)] = 0.$$

Now as the tangent line has $s-$point contact at (t_1), equations (2) and its first $s-1$ derivatives are satisfied for $t=t_1$. The derivative of the next higher order is not zero when $t=t_1$, for if it were the tangent line would have contact of higher order. Substituting $f_1^{(i)}(t_1)$, $f_2^{(i)}(t_1)$, $f_3^{(i)}(t_1)$ for $f_1(t)$, $f_2(t)$, $f_3(t)$, respectively, in equation (2), there results the determinant $F_i(t_1)$ in the expanded form. Hence $F_i(t_1)=0$, $(i=2,3,\ldots, s-1)$, while $F_s(t_1) \neq 0$ for every such point (t_1).

Conversely, if $F_i(t_1)=0$, $(i=2,3,\ldots, s-1)$, $F_s(t_1) \neq 0$, the tangent line at the point given by t_1 has $s-$point contact with the curve.

For, as $F_s(t_1) \neq 0$, not all the two rowed minors of the matrix

$$\left\| \begin{matrix} (f_1(t_1) & f_2(t_1) & f_3(t_1) \\ (f_1'(t_1) & f_2'(t_1) & f_3'(t_1) \end{matrix} \right\|$$

vanish, and hence there exist constants a, b, c, not all zero, such that

$$af_1(t_1)+bf_2(t_1)+cf_3(t_1)=0,$$

$$af_1^{(i)}(t_1)+bf_2^{(i)}(t_1)+cf_3^{(i)}(t_1)=0, \quad i=1,2,\ldots, s-1)$$

Therefore the function, $af_1(t)+bf_2(t)+cf_3(t)$, and its first $s-1$ derivatives vanish for $t=t_1$, which means that the line, $ax_1+bx_2+cx_3=0$, has at least $s-$point contact with the curve at the point (t_1). The line cannot have contact of any higher order, for if it did, $F_s(t_1)$ would be zero, which is contrary to the hypotheses.

Let it now be assumed that equations (1) may be put in the form,

$$x_1 : x_2 : x_3=f_1(T) : f_2(T) : f_3(T),$$

where T is a function of t such that for every value of T there exists at least two values of t. This implies that to each point on the curve corresponds two or more values of the parameter t. Using this representation, it is found that $F_i(t)$ is the determinant

$$|f_1(T) \quad f_2'(T)dT/dt \quad d^if_3(T)/dt^i|,$$

which reduces to

$$|f_1(T) \quad f_2'(T) \quad d^if_3(T)/dt^i|\,dT/dt.$$

Hence all the roots of $dT/dt=0$ are roots of $F_i(t)=0$. When $i=2$ it is readily shown[3] that $F_2(t)=F_2(T)(dT/dt)^3$.

Suppose now that the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ in equations (1) have a common factor $h(t)$. That is, let the equations (1) be written in the form

$$x_1 : x_2 : x_3=h(t)g_1(t) : h(t)g_2(t) : h(t)g_3(t).$$

The determinant $F_i(t)$ for these equations is

$$|h(t)g_1(t) \quad h'(t)g_2(t)+h(t)g_2'(t) \quad d^ih(t)g_3(t)/dt^i|$$

which reduces to $h^2(t)|g_1(t) \; g_2'(t) \; d^ih(t)g_3(t)/dt^i|$. For $i=2$, $F_2(t) =h^3(t)|g_1(t) \; g_2'(t) \; g_3''(t)|$, a well-known relationship.[4] Thus all the roots of $h(t)=0$ are roots of $F_i(t)=0$.

[3]See Muir and Metzler, Theory of Determinants, New York, 1930, P. 662.
[4]See Muir and Metzler, loc, cit., P. 662.

Finally let equations (1) represent a line. The functions $f_1(t)$, $f_2(t)$, $f_3(t)$ are therefore linearly dependent, and hence the Wronskian, $F_2(t)$, vanishes identically. This insures the identical vanishing of $F_i(t)$ for $i>2$. Conversely, if $F_2(t)$ vanishes identically equations (1) represent a line.

In conclusion, suggestions in regard to the usefulness of the above results may be made. First, the identical vanishing of $F_2(t)$ means that the equations (1) represent a straight line. Second, the roots of the equations $F_i(t)=0$, $(i=2,3\ldots)$, may indicate various singularities or forms of the equations (1). Thus if t_1 is a root of the equations $F_i(t)=0$, $(i=2,3,\ldots,s-1)$, and if $F_s(t_1)\neq 0$, the point given by t_1 is a point at which the tangent line has $s-$point contact with the curve.

A multiple root, t_1, of all the equations $F_i(t)=0$, $(i=2,3,\ldots)$, may mean that the functions $f_1(t)$, $f_2(t)$, $f_3(t)$ have a common factor, which is true if $f_1(t_1)=0$, $f_2(t_1)=0$, $f_3(t_1)=0$. Or a single root may mean that to each point on the curve corresponds more than one value of the parameter. This condition may be recognized by investigating the roots of the equation $af_1(t)+bf_2(t)+cf_3(t)=0$. If, in general, two or more values of t give the same point on the curve, the parametric form (1) is not the proper representation of the curve in question. A proper representation may be made by making use of a method due to Lüroth.[5]

A single or multiple root, t_1, of $F_i(t)=0$, $(i=2,3,\ldots)$, may mean that a multiple point of the curve exists which is given by two or more equal values of the parameter. Such a multiple point exists if $f_1(t_1)/f_1'(t_1)=f_2(t_1)/f_2'(t_1)=f_3(t_1)/f_3'(t_1)$. The multiplicity of the root, t_1, of $F_i(t)=0$ for various types of such singularities may be determined most advantageously by making use of the invariant nature of $F_i(t)$. Thus the parametric equations of the curve, which possesses the multiple point to be studied, may be taken in such a way that the multiple point falls at a vertex of the triangle of reference. This was done for the case of the ordinary cusp. The various types of such multiple points are so numerous that space has not been taken in this paper for their study.

[5]See Lüroth, Math Annalen, IX, p. 163.

The Teacher's Department

Edited by

JOSEPH SEIDLIN

TEACHING MATHEMATICS

In teaching mathematics I derive the greatest satisfaction when I feel that my students are deriving the greatest satisfaction. This seems to me to occur when the students are actively engaged in working out things for themselves. The element of personal discovery on their part, which is akin to research activity, enlivens their study and brings best results.

Accordingly, I rarely use the lecture method but employ rather the question method designed to lead them to discover and work out the mathematics by themselves. Rarely are my questions designed merely to test the student's preparation or knowledge of the subject but rather I attempt to be the leader in an exploring party in which all are participants.

E. J. MOULTON,
Northwestern University.

MINUTIAE IN TEACHING PROCEDURE

Every now and then I discover, often accidentally, some small matter of arrangement, or order, or illustration, that seems to "improve" the presentation or development of a topic. Other teachers, interested in their teaching, doubtless experience like "discoveries" thus realizing an improvement and the sad fact that it is a long road to the ideal of perfect teaching.

I have often wished that I might have a stenographer present at all my recitations recording everything that happens. Confronted by such records, I am sure I should be both ashamed and encouraged. Perhaps, too, were such records kept, I could be of some tangible assistance to young teachers, at least in our own department. Possibly such records might obviate the embarrassments and the pitfalls and the wastage (of both teaching and learning hours) common to all "hit and miss" teaching.

Ever since my attention was first drawn to the Teacher's Department in the National Mathematics Magazine, I threatened to suggest

the establishment of some sort of "clearing house" for the cumulative experiences of our colleagues. I have no written evidence, however, with which I could challenge the Editor's priority of that idea. Thus we add one more instance of simultaneity of invention or discovery in the history of our subject.

As a typical illustration of "minutiae" I shall choose an apparently trivial modification in the transition *from* the definition and numerical illustration of a determinant of the second order *to* its application of the solution of a pair of simultaneous equations.

Exhibit A

Definition:

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

Numerical Illustrations:

$$\begin{vmatrix} 3 & 2 \\ 4 & 5 \end{vmatrix} = 3\cdot 5 - 2\cdot 4 = 7$$

$$\begin{vmatrix} 1 & 6 \\ -2 & -5 \end{vmatrix} = 1\cdot -5 - 6\cdot -2 = -5 + 12 = 7, \text{ etc.}$$

Application:

$$2x + 3y = 7$$
$$3x + 2y = 8$$

(1) Solve for x by "eliminating" y

(2) Solve for x "with the aid of determinants"

Exhibit B

Definition:

As in *A*

Numerical Illustrations (if any):

As in *A*

Application:

$$a_1x + a_2y = a_3$$
$$b_1x + b_2y = b_3$$

(1) Solve for x by "eliminating" y

(2) Solve for x "with the aid of determinants"

Exhibit C

Definition:

As in *A*

Numerical Illustrations:

$$\begin{vmatrix} 3 & 2 \\ 5 & 7 \end{vmatrix} = 3\cdot 7 - 2\cdot 5 = 21 - 10 = 11$$

$$\begin{vmatrix} 4 & 2 \\ 6 & 7 \end{vmatrix} = 4\cdot 7 - 2\cdot 6 = 28 - 12 = 16$$

Applications:

$$3x + 2y = 4$$
$$5x + 7y = 6$$

(1) Solve for x by "eliminating" y

$$x = \frac{4\cdot 7 - 2\cdot 6}{3\cdot 7 - 2\cdot 5} = \frac{28 - 12}{21 - 10} = \frac{16}{11}$$

(2) Solve for x "with the aid of determinants"

Now, there seem to be no striking differences between the three exhibits and yet this is what happens: Teaching it, as per Exhibit A, the instructor "puts it across" with rarely an interruption from the class; teaching it, as per Exhibit B, the instructor may get a little encouragement and assistance from one or two of the brightest students; teaching it, as per Exhibit C, the instructor finds no need of "developing" (2). Most of the class is too eager and thrilled and willing to suggest the solution "with the aid of determinants."

It requires no complicated or involved experimental set-up to verify the above statement. Indeed it is a very simple matter if one teaches in a school "blessed" with three or more sections in college algebra.

May I make clear the nature of this (my) contribution to teaching procedures. It is NOT to suggest a scheme for introducing determinants (I hope, and I am quite certain, that most experienced teachers present the topic at least as effectively as suggested in Exhibit C). I merely wish to emphasize that apparently trivial matters of arrangement of choice of illustration are as much a live issue in the improvement of teaching, our own teaching, as generalities, albeit glittering generalities. Also, in other and plainer words, most of us could improve our teaching through some available record of the cumulative experience of our colleagues. —O.

Mathematical Notes

Edited by

L. J. ADAMS and I. MAIZLISH

The Department of Mathematical Notes devotes space this month to the publication of a list of the articles and notes of the late Dr. Raymond Garver, of the University of California at Los Angeles and member of the Editorial Board of the National Mathematics Magazine. This is done with the hope that young research workers will receive lasting inspiration and encouragement from this mute evidence of the remarkable output of a man who passed away in the thirty-fifth year of his life.

In the short span of less than a decade, Dr. Garver made a prodigious contribution to the extension of the frontiers of mathematical knowledge. An examination of the list shows that his work reached the mathematical society of the world. The dates of publication reveal the intensity of his activity.

Surely many of the younger readers of this magazine, and the older ones, too, will marvel at the monument Dr. Garver has left, and will be actuated to renewed energy in exploring the boundaries of mathematical thought.

1. The Analyst, 1874-1933, Scripta Mathematica, Vol. 1 (1933), pp. 247-251, 322-326.
2. An Application of the Theorem of Junker, Tôhoku Math. Jour., Vol. 33 (1931), pp. 260-264.
3. Bieberbach's Trisection Method, Scripta Mathematica, Vol. 3 (1935), pp. 251-255.
4. Binomial Quartic as a Normal Form, Bull. Amer. Math. Soc., Vol. 33 (1927), 677-680.
5. Concerning Polynomial Functions with Certain Properties, Annals of Math., Ser. 2, Vol. 31 (1930), pp. 366-370.
6. Concerning the Limits of a Measure of Skewness, Annals of Mathematical Statistics, Vol. 3 (1932), pp. 358-360.
7. Concerning Two Square Root Methods, Calcutta Math. Soc. Bulletin, Vol. 24 (1932), 99-102.
8. A Definition of Group by Means of Three Postulates, Amer. Jour. of Math., Vol. 57 (1935), pp. 276-280.
9. Determinants and the Roots of an Equation, Jour. Indian Math. Soc., Vol. 19 (1932), pp. 156-160.
10. Division Algebras of Order Sixteen, Annals of Math., (2), Vol. 28 (1927), pp. 493-500.
11. The Edgeworth Taxation Phenomemon, Econometrica, Vol. 1 (1933), pp. 402-407.

12. Effect of Taxation on a Monopolist, Amer. Econ. Review, Vol. 22 (1932), pp. 463-465.
13. Enrique Cruchaga's Solution of the Quartric Equation, Amer. Math. Monthly, Vol. 37 (1930), pp. 303-304.
14. Error Expressions for Certain Continued Fractions, Bull. Amer. Math. Soc., Vol. 39 (1933), pp. 137-141.
15. The Gauss-Lucas Theorem, Mathematical Gazette, Vol. 16 (1932), page 337.
16. Invariantive Aspects of a Transformation on the Brioschi Quintic, Annals of Math., (2), Vol. 32 (1931), pp. 478-484.
17. Linear Fractional Transformations on Quartic Equations, Amer. Math. Monthly, Vol. 36 (1929), pp. 208-212.
18. Mathematical Induction, Mathematics Teacher, Vol. 26, (1933), pp. 65-69.
19. Mathematics of Small Loans, Amer. Econ. Review, Vol. 21 (1931), pp. 693-695.
20. Mathematics of Small Loans, Correction, Amer. Econ. Review, Vol. 22 (1932), pp. 269-270.
21. A New Normal Form for Quartic Equations, Bull. Amer. Math. Soc., Vol. 34 (1928), 310-314.
22. A Note Concerning a Transformation of the Brioschi Quintic, Tôhoku Math. Jour., Vol. 35 (1932), pp. 253-256.
23. Note Concerning Group Postulates, Bull. Amer. Math. Soc., Vol. 40 (1934), pp. 698-701.
24. A Note on Bieberbach's Trisection Method, Journal Für Mathematik, Vol. 173 (1935), pp. 243-244.
25. Note on Partial Fractions, Amer. Math. Monthly, Vol. 34 (1927), pp. 319-320.
26. Note on Square Roots, Mathematical Gazette, Vol. 16 (1932), pp. 339-340.
27. Notes Mathematiques, Mathesis, 41 (1927), pp. 411-413.
28. On the Approximate Solution of Certain Equations, Amer. Math. Monthly, Vol. 39 (1932), pp. 476-478.
29. On the Brioschi Normal Quintic, Annals of Math., (2), Vol. 30 (1929), pp. 607-612.
30. On the Discriminant of the Cubic Equation, School Science and Mathematics, Vol. 29 (1929), pp. 474-476.
31. On the Evaluation of a Certain Type of Property, Accounting Review, Vol. 7 (1932), 70-74.
32. On the Reduction of the General Quartic to Binomial Form, Amer. Math. Monthly, Vol. 37 (1930), pp. 245-248.
33. On the Removal of Four Terms from an Equation by Means of a Tschirnhaus Transformation, Bull. Amer. Math. Soc., Vol. 35 (1929), pp. 73-78.
34. On the Theorems of Pappus, School Science and Mathematics, Vol. 27 (1927), pp. 937-940.
35. On the Transformation which leads from the Brioschi Quintic to a General Principal Quintic, Bull. Amer. Math. Soc., Vol. 36 (1930), pp. 115-119.
36. Perfect Non-Dense Points Set, Amer. Math. Monthly. Vol. 34 (1927), pp. 36-37.
37. Caliban's Problem Book, A Review, Nat. Math. Magazine, Vol. 9 (1935), pp. 213-215.

38. Postulates for Special Types of Groups (probable title), soon to appear in Bull. Amer. Math. Soc., 1936, Number 2 or 4.
39. Proof of a Theorem of Brioschi, Messenger of Math., Vol. 58 (1928), pp. 24-26.
40. Quartic Equations with Certain Groups, Annals of Math., (2), Vol. 30 (1928), 47-51.
41. Quartic Equations with the Alternating Group, Tôhoku Math. Jour., Vol. 32 (1930), pp. 306-311.
42. Rational Normal Form for Certain Quartics, Bull. Amer. Math. Soc., Vol. 34 (1928), 73-74.
43. A Reading List in the Elementary Theory of Equations, Amer. Math. Monthly, Vol. 40 (1933), pp. 77-84.
44. Solution of the Quartic Equation, Amer. Math. Monthly, Vol. 35 (1928), 558-560.
45. A Square Root Method and Continued Fractions, Amer. Math. Monthly, Vol. 39 (1932), pp. 533-535
46. Sur l'equation du Troisiéme degré, Mathesis, Vol. 42 (1928), pp. 344-346.
47. Sur le Nombre π, Mathesis, Vol. 41 (1927), pp. 411-413.
48. Sur les Transformations homographiques, Mathesis, Vol. 42 (1928), pp. 163-165.
49. Transformations of One Principal Equation into Another, Annals of Math., (2), Vol. 28 (1927), pp. 112-116.
50. The Transformation of $y=f'(x)$, Mathematical Gazette, Vol. 17 (1933), pp. 251-256.
51. Transformations on Cubic Equations, Amer. Math. Monthly, Vol. 36 (1929), pp. 366-369.
52. The Tschirnhaus Transformation, Annals of Math., (2), Vol. 29 (1928), pp. 319-333.
53. Tschirnhaus Transformations in the Elementary Theory of Equations, Amer. Math. Monthly, Vol. 38 (1931), pp. 185-188.
54. Tschirnhaus Transformations on Certain Rational Cubics, Amer. Math. Monthly, Vol. 34 (1927), 521-525.
55. Two Notes on Cyclic Cubics, Amer. Math. Monthly, Vol. 35 (1928), 435-436.
56. Type of Function with K Discontinuities, Amer. Math. Monthly, Vol. 34 (1927), pp. 362-363.
57. The Solution of Problems in Maxima and Minima by Algebra, Amer. Math. Monthly, Vol. 42 (1935), pp. 435-437.
58. An Evaluation of a Certain Double Integral, Math. News Letter, Vol. 8 (1933), pp. 38-40. (November).
59. On the Roots of a Cubic and Those of Its Derivative, Math. News Letter, Vol. 6 (1932), pp. 24-27. (April-May).
60. The Approximation of Real Roots of Equations by Simple Continued Fractions, Math. News Letter, Vol. 7 (1932), pp. 20-22. (November).
61. Compound Interest, Math. News Letter, Vol. 7 (March, 1933), pp. 3-8.
62. On the Nature of the Roots of a Quartic Equation, Math. News Letter, Vol. 7, (January, 1933), pp. 7-8.
63. On the Relative Accuracy of Simpson's Rules and Weddle's Rule, Amer. Math. Monthly, Vol. 34 (1927), p. 369.

64. A Note on the Function $y=x^x$, Amer. Math. Monthly, Vol. 34 (1927), p. 429.
65. A Normal Form for Certain Quartics, Messenger of Mathematics, Vol. 56 (1927), pp. 184-186.
66. Three Transformations on Quartic Equations, Messenger of Math., Vol. 57 (1927), pp. 99-101.
67. Formulas for Partial Franctions, School Science and Mathematics, Vol. 28 (1928), pp. 614ff.

The compilation of this list is the work of Professor W. M. Whyburn, a colleague of Dr. Garver at the University of California at Los Angeles.

Dr. Harry Kirkpatrick has been made a member of the Department of Mathematics of Occidental College, Los Angeles, California. Dr. Kirkpatrick has been connected with the University of Hawaii for the past four years.

Perhaps the most extensive collection of mathematical models is that of the firm of Martin Schilling in Leipzig, Germany. Their one hundred page catalogue is instructive as well as descriptive.

Professor W. V. Parker of the Georgia School of Technology informs us of the following changes in the Department of Mathematics at that institution:

Associate Professor H. K. Fulmer is on leave of absence studying at Cornell University.

Assistant Professor R. A. Hefner has been promoted to an associate professorship.

Dr. D. H. Ballou and Dr. F. H. Steen have been promoted to assistant professorships.

Messrs. G. A. Rosselot, W. H. Sears, R. F. Watkins and A. L. Starratt have been appointed instructors.

At the University of Chicago, Dr. Ralph Hull has been appointed instructor in mathematics for the year 1935-36.

Professor Edgar E. DeCou, head of the Mathematics Department at the University of Oregon, informs us that there has been an increase of 30% in the registration of students in his department for the current scholastic year. Many of the new students are social science majors. This is additional evidence of the increased use of mathematics by workers in that branch of human knowledge and endeavor.

Professor Decou also reports these additions to his staff:

Kenneth S. Ghent, Ph.D. (University of Chicago), instructor.

Mrs. Lou Moursund, M. A. (Brown University), instructor.

He further states that at his request, as Permanent Secretary of Pi Mu Epsilon at the University of Oregon, the national officers extended their chapter to include Oregon State College as a joint chapter. At least one joint meeting each year is planned.

The following have been invited to deliver lectures before The International Congress of Mathematicians at Oslo on July 13-18, 1936: Ahlfors, Banach, Birkhoff, E. Cartan, van der Corput, Fréchet, Gelfond, Hasse, Hecke, Knintchine, Landau, Neugebauer, J. Nielson, Ore, Oseen, Siegel, Skolem, Strmer, Veblen and N. Wiener. The addresses will constitute a general outline of modern mathematics.

The Mathematical Association of England maintains a *Bureau for the Solution of Problems.* All members of the association are invited to send problems to this bureau for solution. The Honorable Secretary of the Bureau is A. S. Gosset Tanner.

The April, 1935 number of the *Japanese Journal of Mathematics* contains two hundred brief abstracts of all mathematical papers published in Japan during the year 1934. Reserach workers will find these indexed abstracts useful and time-saving.

The American Mathematical Society announces the following spring and summer meetings:

1. New York City. February 29. Professor E. Hopf has been invited to deliver a lecture on *Some Metrically Transitive Systems.*
2. New York City. April 10-11.
3. Chicago, Illinois. April 10-11.
4. Berkeley, California. April 10-11.
5. Cambridge, Massachusetts. Summer Meeting and Colloquium. August 31-September 5. In connection with the Harvard Tercentenary.

The Topological Congress met in Moscow September 4-10, 1935. There were thirty-seven delegates, including ten from America. The meetings were devoted entirely to the recent research in Topology.

The Scientific Congress met in Mexico City September 8-17, 1935. There were thirty delegates from the United States. Papers presented by Americans were:

A Survey of Statistical Means. E. L. Dodd.

An Elementary Determination of a Rigorous Equation for the Figure of the Earth. (Illustrated with lantern slides). E. V. Huntington.

A Theory of Cosmic Radiation and Its Experimental Test. M. S. Vallarta.

At this congress in Mexico City, a resolution was adopted urging the adoption of the World Calendar, a certain form of the thirteen month reform calendar.

Dr. G. A. Linhart of Riverside, California has found that a large variety of natural processes follow the law

$$y = y_{\infty} kx^{K}/(1+kx^{K})$$

Some of his more recent investigations have been with the data of the Experimental Station, a branch of the University of California at Riverside, California.

Problem Department

Edited by

T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

Send all communications about problems to T. A Bickerstaff, University, Mississippi.

SOLUTIONS

No. 45. Proposed by C. D. Smith, Mississippi State College.

Show that the Brocard points of a triangle may be taken as special cases of Miquel points.

Solution by *C. A. Balof*, Lincoln College, (Illinois).

Let ABC be the given triangle, and M be the Brocard point for which angles MAB, MBC, and MCA are equal. Then M is the point of intersection of three circles: c_1,c_2, and c_3, such that c_1 is tangent to BC at B and passes through A, c_2 is tangent to CA at C and passes through B, and c_3 is tangent to AB at A and passes through C. Hence M is a Miquel point for points ABC with respect to triangle ABC. For, if on AB, BC, and CA, we mark points B, C, and A respectively, c_1, c_2, and c_3 are circles through each vertex of triangle ABC and the marked points on the adjacent sides.

It may be noted in this connection, that if three points are marked, one on each side of a triangle but not at a vertex, there is a single Miquel point for the marked points with respect to the triangle. But for the vertices of the triangle, every point in the plane of the triangle and not on one of its sides is a Miquel point.

No. 92. Proposed by W. V. Parker, Georgia Tech.

Prove that all triangles of maximum area inscribed in the ellipse $x^2/a^2+y^2/b^2=1$ envelope the ellipse $x^2/a^2+y^2/b^2=\frac{1}{4}$.

Solved by *J. Rosenbaum*, Hartford, Conn.

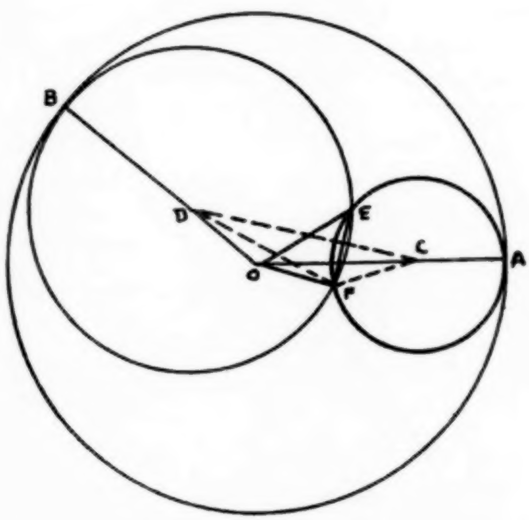
Using the notation of solutions to Nos. 89 and 91, the distance from 0, of a side such as $P'X'$ of triangle $P'X'Y'$ is $a/2$, hence the sides of all the equilateral triangles in C_1 envelope the circle C_3 whose center is 0, and radius $a/2$, so that the projection of all these sides will envelope the projection of C_3 (because tangency is preserved), which projection is easily seen to be the last ellipse.

Also solved by *A. C. Briggs*, Wilmington, Ohio.

No. 94. Proposed by W. B. Clarke, San Jose, Cal.

OA and OB are two radii of a circle and not in a straight line. On OA take OC, with C inside circle and not at midpoint of OA. On OB take BD = OC. With centers C and D and radii CA and BD, describe circles intersecting at E and F, taking E as intersection that lies within angle AOB. Show that angle EFO is a right angle.

Solution by *R. A. Miller*, State University of Iowa, Iowa City.



Given: Circle O, AO, OB radii not in straight line. OC = BD. (D) and (C) with radii BD and CA intersect at F.

To Prove: $EFO = 90°$

Draw CF, FD, CD

Proof: $DF = BD = OC$

$CF = AC = OD$

$\triangle OCF \equiv \triangle ODF$

$\angle ODF = \angle OCF$

$\triangle FDC \equiv \triangle DOC$

$\angle FDC = \angle OCD$

$\therefore \angle OCF + \angle OCD = \angle ODF + \angle FDC$

$\angle FCD = \angle ODC$

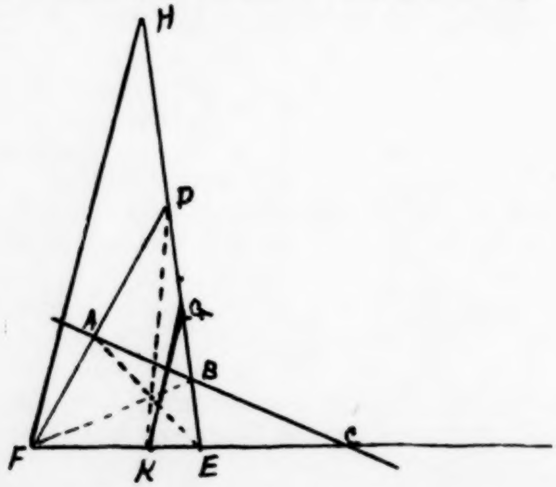
$\therefore$ CFOD is isosceles trapezoid, and CD is parallel to FO.

Since EF is radical axis of (C) and (D), it is bisected at right angles by CD.

$\therefore OF \perp EF$

No. 104. Proposed by Walter B. Clarke, San Jose, Cal.

Three lines, a, b, and c, which are neither concurrent nor parallel intersect a fourth line XY in the points A,B, and C, respectively. Lines a and b intersect at D; b and c intersect at E; and a and c intersect at F. On ED, take $EG = CE$ and $GH = CF$ ($EH > EG$). Through G, draw a parallel to FH cutting FC at K. Show that FB, AE, and KD, are concurrent regardless of which way EG is taken.



Solved by *Alta H. Samuels*, L. S. U.

The above figure illustrates the conditions stated in the given problem. In $\triangle$ DFE,

(1) $DA \cdot FC \cdot BE = AF \cdot EC \cdot BD$ (Altshiller-Court, § 225) or
(2) $DA \cdot BE = (AF \cdot EC \cdot BD)/FC = (AF \cdot BD)GE/HG$ ($EC = GE$ and $FC = GH$). In $\triangle$ HFE

(3) $GE/HG = KE/FK$ (GK is drawn parallel to HF)

Substituting (3) in (2) and multiplying by FK gives

$$DA \cdot FK \cdot BE = AF \cdot KE \cdot BD$$

Thus the points A, B, and K divide the sides of $\triangle$ DFE into six segments such that the product of three segments having no common end is equal to the product of the remaining three segments, therefore, the lines AE, BF and KD are concurrent. (Altshiller-Court, § 238.)

If EG is taken in the opposite direction, EH must be taken in that direction, also, since $EH > EG$. Therefore the triangle HFE would fall on the opposite side of line FC, but equations (1), (2) and (3) would still hold true, since in these equations, magnitudes only, and not directions, are considered.

Also solved by *A. C. Briggs, Henry Schroeder* and *C. A. Balof.*

PROBLEMS FOR SOLUTION

No. 114. Proposed by W. C. Janes, Kansas State College.

If P is a point in one face of a parallelopiped and Q is a point in the opposite face, find the shortest path (lying in the faces of the parallelopiped) from P to Q.

No. 115. Proposed by G. W. Wishard, Norwood, Ohio.

Prove that every odd square in the octonary system (scale of eight) ends in 1, and if this 1 be cut off, the remaining part is a triangular number.

No. 116. Proposed by Walter B. Clarke, San Jose, Calif.

I is the incenter, J and K are any two of the excenters of a right triangle. Let perpendiculars from J and K to the nearest sides of the triangle intersect at N. Show that I and N are equi-distant from the hypotenuse.

Book Reviews

Edited by
P. K. SMITH

A Comment on "Early American Arithmetics". E. R. Sleight, October, 1935.

The first paragraph of this article contains a statement to the effect that it is generally believed that Isaac Greenwood was the author of the first arithmetic to be written by an American. It says also that there seem to be only three copies now in existence—two in the library at Harvard and one in the Congressional library.

It may be of interest to the readers of the NATIONAL MATHEMATICS MAGAZINE to know that conclusive evidence of the authorship of this work was located by the writer in an advertisement whose earlier location would have prevented any dispute over the matter. The advertisement in the (Boston) *Weekly News-Letter*, 29 May, 1729, runs as follows: "Just Published *Arithmetick Vulgar and Decimal*; *with the Application thereof to Variety of Cases in Trade and Commerce.* By Isaac Greenwood, A. M. Hollisian Professor of the Mathematicks, and Philosophy. To be Sold by Thomas Hancock ât the Bible & Three Crowns near the Town Dock, Boston." (Repeated 5 and 12 June.) This evidence was published first in my *Introduction of Algebra into American Schools in the Eighteenth Century,* 1924, p. 68.

A further note on "Isaac Greenwood's Arithmetic" is found in *Scripta Mathematica,* March, 1933, p. 262-263p. This Note adduces evidences of the use of Greenwood's text from copies of it examined at the Massachusetsts Historical Society, Watkinson Library in Hartford, Boston Public Library and Yale University library. No attempt at a thorough search has been made but it is altogether probable that there are copies in a number of other libraries.

LAO G. SIMONS,
Hunter College of the City of New York.

A First Course in Differential Equations. By Norman Miller, Oxford University Press, 1935. 148 pages.

According to the preface, "This course is intended for students who have taken a first course in calculus and presupposes some familiarity with the processes of algebra and analytic geometry. It aims

to give a concise and clear account of the most useful methods of solution of differential equations and to provide ample material for practice in these methods. At the same time the student is kept in close touch with those applications of the subject to which it owes much of its development."

The book contains twelve chapters and a brief historical note of four pages. The first chapter deals with definitions, geometric interpretation, derivation of ordinary differential equations and boundary conditions. The second chapter gives the usual classic methods of integrating equations of the first order and first degree. The third and fourth chapters treat equations of the first order but not of the first degree and their singular solutions. The fifth chapter takes up applications of first order differential equations with accent on laws of growth, rate problems, geometric problems, trajectories and mechanical and phsyical application. The sixth chapter gives a discussion of linear equations with constant coefficients and D operators. The seventh chapter is devoted to homogeneous linear equations, exact linear equations and series solutions. The eighth chapter gives special methods for solving second and higher order equations, including among other methods variation of parameters. In Chapter IX appear geometric and mechanical problems involving second order differential equations, with one article on harmonic motion and resonance. Chapter X is devoted to systems of ordinary differential equations and total differential equations. Chapter XI gives an exposition of the classic methods of Lagrange and of Lagrange and Charpit for first order partial differential equations and of the methods of formation of partial differential equations. Chapter XII gives a brief indication of some of the methods used in partial differential equations of order higher than the first with physical applications.

Naturally it falls to the lot of the reviewer to make some criticisms. In this particular case he can make only a very few. If, in the illustrative example of page 8, the symbol $(1+y'^2)^{3/2}$ means the cube of the positive square root of $1+y'^2$ then the general solution (for r positive) of $(1+y'^2)^{3/2}=r\,d^2y/dx^2$ should be $y=b-\sqrt{r^2-(x-a)^2}$, that is, the portion of the circle of center (a, b) and radius r which is concave up if the y axis is oriented upward. In Chapter V, Article 25, the normal and tangent distances are essentially positive and numerical values should be indicated; compare with example 1, page 84, where attention is given to this question of signs. There should appear a plus or minus sign in example 1, page 39. On page 43, line 6, velocity should appear instead of speed. In Chapter XI it might be well to

indicate that the standard forms given are special cases where the Lagrange-Charpit methods afford an integral immediately.

The author is to be especially complimented on the clear-cut distinction made in Chapter III between the various roles played by the symbol p in the equation $f(x,y,p)=0$, and on the very careful statement of problems in Chapter IX.

The book as a whole is as clear and concise as the author indicated in his preface; there is also a good collection of problems at the end of each chapter in addition to those appearing immediately after each new method. It certainly deserves careful inspection by instructors in differential equations courses.

WILLIAM E. BYRNE,
November 29, 1935.